

# Quantifier-free least fixed point functions for phonology

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MIT

# Introduction

- What kind of functions are phonological UR-SR maps?
- Automata-theoretic characterizations have focused on **subsequentiality** (Heinz and Lai, 2013; Payne, 2017; Chandlee and Heinz, 2018)
- **Logical** characterizations of *sets* provide representation-independent complexity hypotheses
- No previous logical characterizations of *functions* approach subsequentiality

- Chandlee and Lindell (forthcoming) capture **input strictly-local (ISL)** functions with **quantifier-free (QF)** logic
- We generalize this with **least fixed-point** extension of QF functions (QFLFP)
- QFLFP offers recursive, *output-based* definitions of functions
- This is a (proper?) subclass of the subsequential functions that tightly fits the typology of phonological functions

# Logical definitions of functions

$\times_1$	$a_2$	$b_3$	$b_4$	$a_5$	$b_6$	$\times_7$
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- **Model** of a string over  $\Sigma$ :

- $D = \{1, 2, \dots, n\}$

$$D = \{1, 2, 3, 4, 5, 6, 7\}$$

- $P_\sigma \subseteq D$  for each  $\sigma \in \Sigma, \times, \times$

$$P_b = \{3, 4, 6\}$$

- A predecessor function  $p$

$$p(2) = 1, \text{ etc.}$$

$\bowtie_1$	$a_2$	$b_3$	$b_4$	$a_5$	$b_6$	$\bowtie_7$
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- **QF logic** of strings:
  - **Terms** are
    - variables  $x, y, \dots, z$  range over  $D$
    - $p(t)$  for term  $t$
  - $P_\sigma(t)$  for  $\sigma \in \Sigma, \bowtie, \bowtie$  and term  $t$

$\times_1$	$a_2$	$b_3$	$b_4$	$a_5$	$b_6$	$\times_7$
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  - $P_\sigma(t)$  for  $\sigma \in \Sigma, \times, \times$  and term  $t$
  - E.g.,  $P_b(x)$

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    - $p(t)$  for term  $t$
  - $P_\sigma(t)$  for  $\sigma \in \Sigma, \times, \times$  and term  $t$
  - E.g.,  $P_b(x), P_b(p(x))$

$\times_1$	$a_2$	$b_3$	$b_4$	$a_5$	$b_6$	$\times_7$
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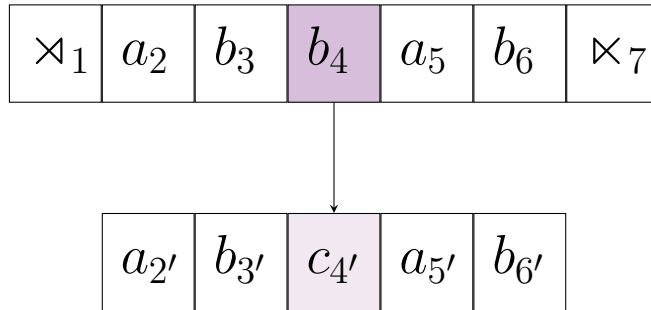
- **QF logic** of strings:

- Syntax:

$$P_\sigma(t) \mid \neg\varphi \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \varphi \rightarrow \psi$$

- E.g.,  $P_b(x) \wedge P_b(p(x))$





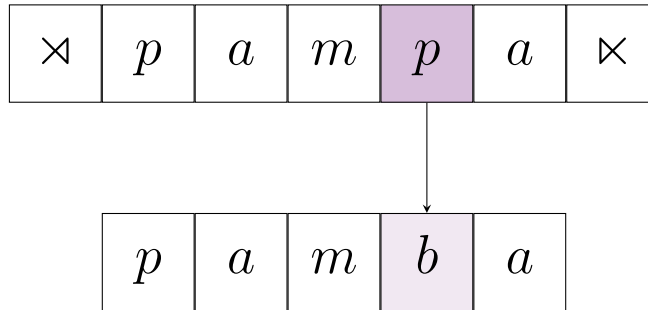
- A **logical transduction** defines an output structure in the logic of the input structure (Courcelle, 1994; Courcelle et al., 2012)

$$P'_a(x) \stackrel{\text{def}}{=} P_a(x)$$

$$P'_b(x) \stackrel{\text{def}}{=} P_b(x) \wedge \neg(P_b(p(x)))$$

$$P'_c(x) \stackrel{\text{def}}{=} P_b(x) \wedge (P_b(p(x)))$$

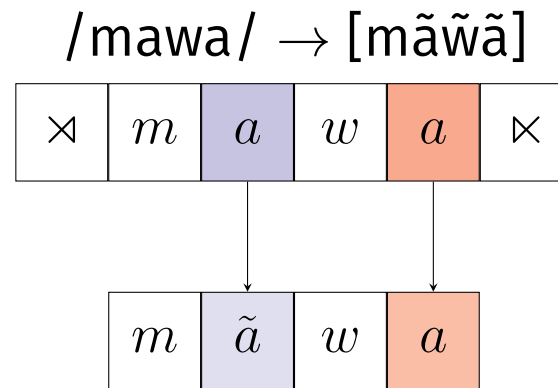
- $b \rightarrow c / b\_$



- **Chandlee and Lindell (forthcoming):** QF transductions capture ISL functions (Chandlee, 2014; Chandlee and Heinz, 2018)

$$\begin{aligned}
 P'_a(x) &\stackrel{\text{def}}{=} P_a(x) \\
 P'_m(x) &\stackrel{\text{def}}{=} P_m(x) \\
 P'_p(x) &\stackrel{\text{def}}{=} P_p(x) \wedge \neg(P_p(p(x))) \\
 P'_b(x) &\stackrel{\text{def}}{=} P_p(x) \wedge (P_m(p(x)))
 \end{aligned}$$

- Long-distance patterns are not QF
- Iterative spreading, e.g. nasal spread in Malay (Onn, 1980)

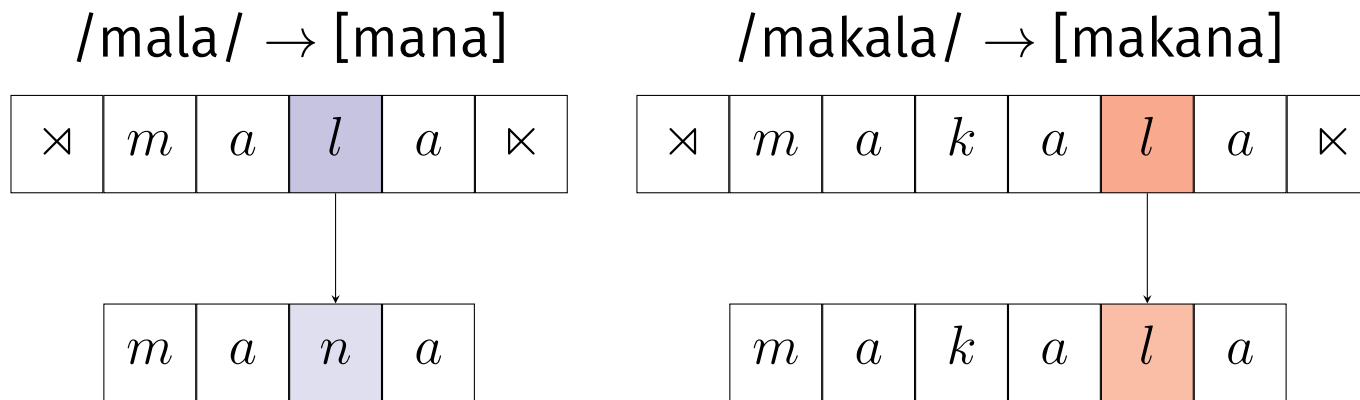


$$P'_{\tilde{a}}(x) \stackrel{\text{def}}{=} P_a(x) \wedge \text{nasal}(p(x))$$

$$P'_a(x) \stackrel{\text{def}}{=} P_a(x) \wedge \neg \text{nasal}(p(x))$$

- $\text{nasal}(x) \stackrel{\text{def}}{=} P_m(x) \vee P'_{\tilde{a}}(x) \vee P'_a(x)$

- Long-distance patterns are not QF
- L-D harmony, e.g. nasal harmony in Kikongo (Ao, 1991)



$$P'_n(x) \stackrel{\text{def}}{=} P_n(x) \vee (P_l \wedge \neg \text{nasal}(p(p(x))))$$

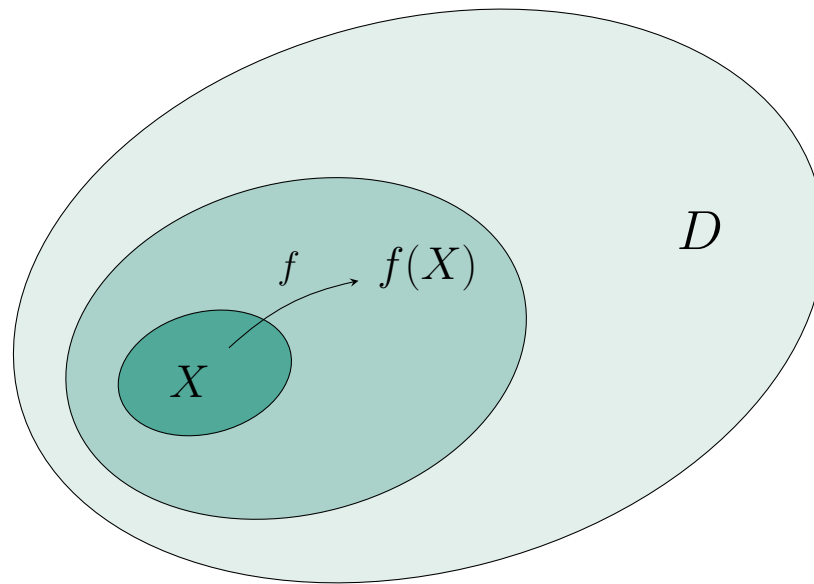
$$P'_l(x) \stackrel{\text{def}}{=} P_l(x) \wedge \neg \text{nasal}(p(p(x)))$$

- $\text{nasal}(x) \stackrel{\text{def}}{=} P_m(x) \vee P_n(x)$

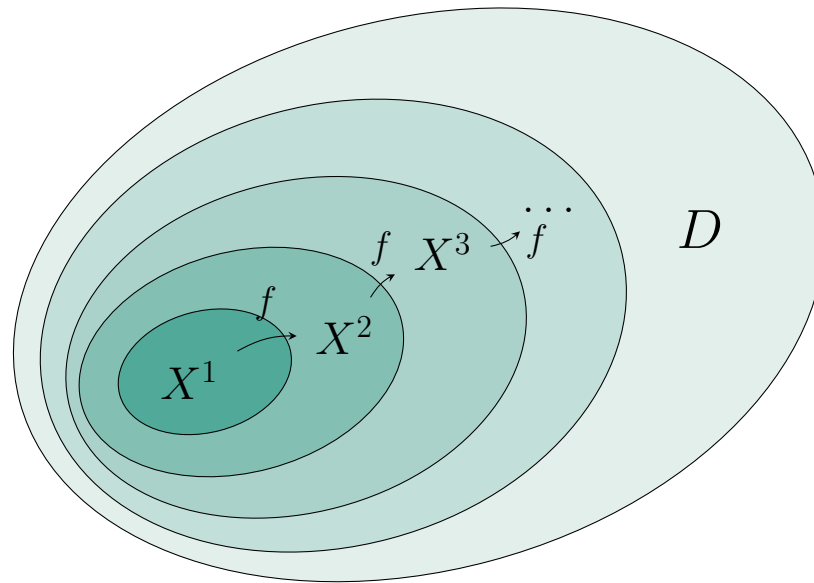
- **Least-fixed point** logic allows:
  - reference to output structures;
  - definition of precedence from predecessor ( $p$ )
- Restriction to QF keeps logic weak

## Least fixed point logic

- An **operator** on  $D$  is a function  $f : \mathcal{P}(D) \rightarrow \mathcal{P}(D)$



- The **least fixed point** of  $f$  is  $\text{lfp}(f) = \bigcup_i X^i$ , where  
 $X^0 = \emptyset, X^{i+1} = f(X^i)$



- $\varphi(A, x)$  with a special predicate  $A(x)$  induces an operator

$$f_\varphi(X) = \{d \in D \mid \varphi(A, x) \text{ is satisfied with } A \mapsto X, d \mapsto x\}$$

### Example

$\times_1$	$a_2$	$b_3$	$a_4$	$a_5$	$a_6$	$c_7$	$a_8$	$\times_9$
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$$\varphi(A, x) = P_a(x) \wedge (P_b(p(x)) \vee A(p(x)))$$



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$$\varphi(A, x) = P_a(x) \wedge (P_b(p(x)) \vee A(p(x)))$$

$$f_\varphi(\emptyset) = \{4\}$$

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$$\varphi(A, x) = P_a(x) \wedge (P_b(p(x)) \vee A(p(x)))$$

$$f_\varphi(\emptyset) = \{4\} \quad X^1$$

$$f_\varphi(\{4\}) = \{4, 5\} \quad X^2$$

$$f_\varphi(\{4, 5\}) = \{4, 5, 6\} \quad X^3$$

$$f_\varphi(\{4, 5, 6\}) = \{4, 5, 6\} \quad X^4 = X^5 = \dots$$

$$\text{lfp}(f_\varphi) = \{4, 5, 6\}$$

- $\varphi(A, x)$  with a special predicate  $A(x)$  induces an operator  
 $f_\varphi(X) = \{d \in D \mid \varphi(A, x) \text{ is satisfied with } A \mapsto X, d \mapsto x\}$

- QFLFP is QF extended with predicates of the form

$$[\text{lfp} \varphi(A, x)](x)$$

for some  $\varphi(A, x)$  in QF extended with  $A(x)$

$$[\text{lfp} P_a(x) \wedge (P_b(p(x)) \vee A(p(x)))](x)$$

$\times_1$	$a_2$	$b_3$	$a_4$	$a_5$	$a_6$	$c_7$	$a_8$	$\times_9$
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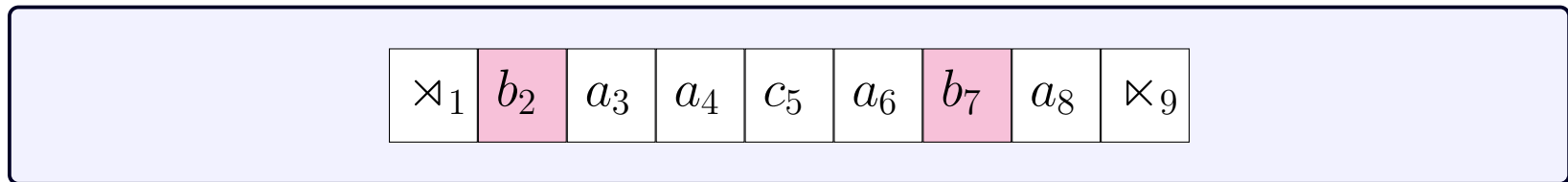
## Example: iterative spreading with blocking

baaa  $\mapsto$  bbbb

baaca  $\mapsto$  bbbca

baacaba  $\mapsto$  bbbcabbb

$$P'_b(x) \stackrel{\text{def}}{=} [\text{lfp}(P_b(x) \vee (A(p(x)) \wedge \neg P_c(x)))](x)$$



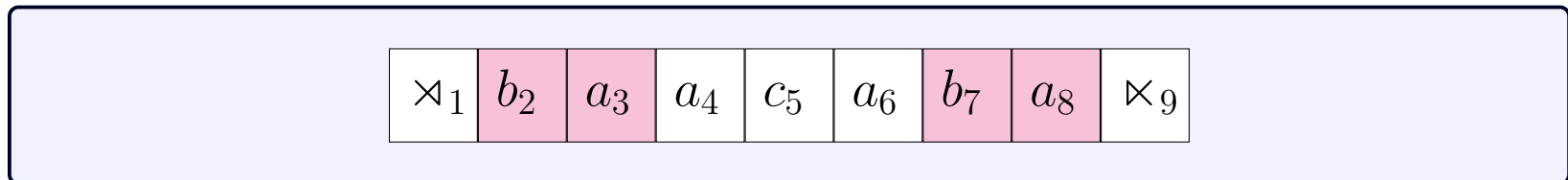
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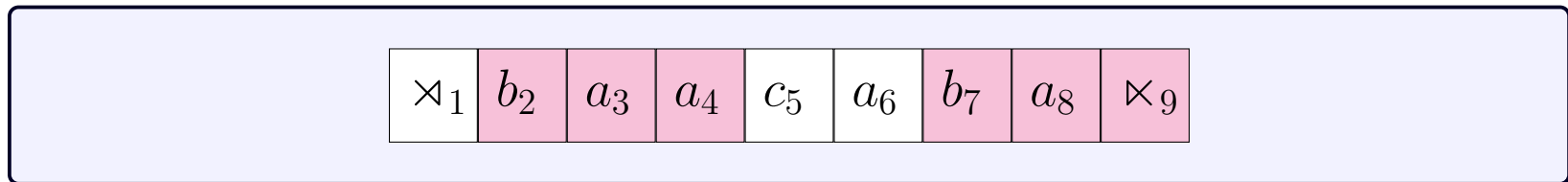
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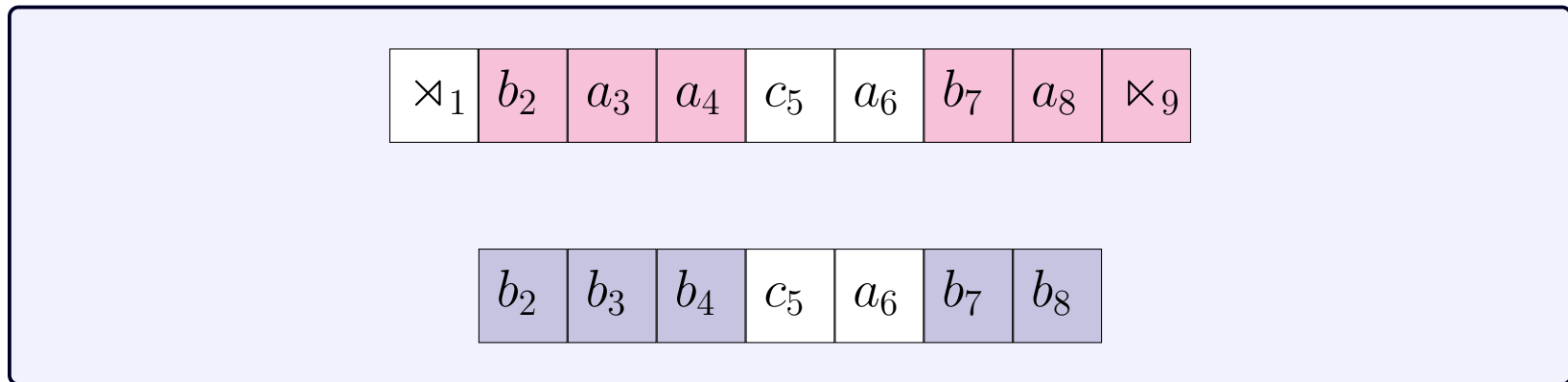
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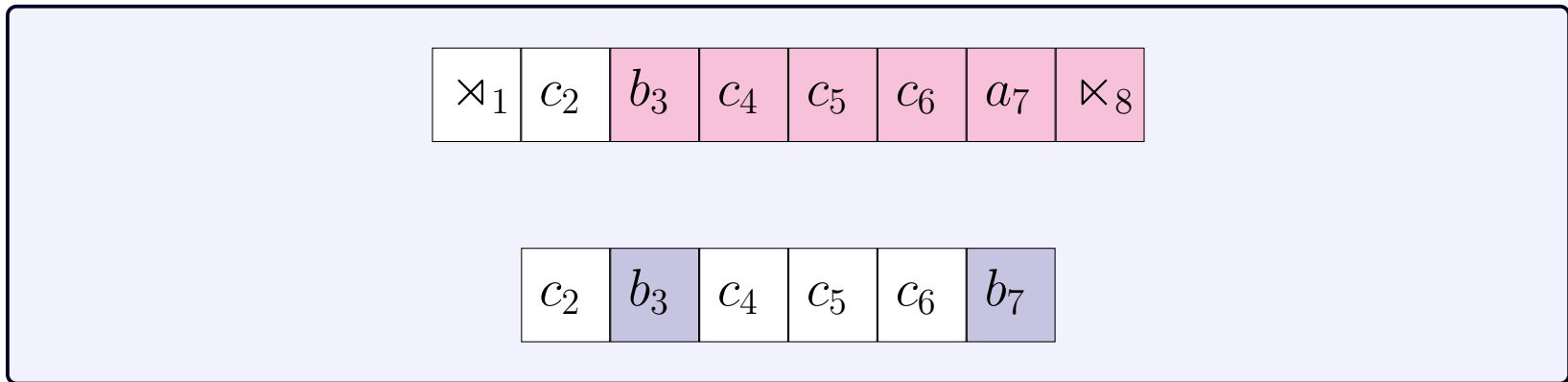
$$P'_b(x) \stackrel{\text{def}}{=} [\text{lfp}(P_b(x) \vee (A(p(x)) \wedge \neg P_c(x)))](x)$$



## Long-distance agreement

cbccca  $\mapsto$  cbcccb

$$P'_b(x) \stackrel{\text{def}}{=} [\text{lfp}(P_b(x) \vee A(p(x)))](x) \wedge \neg P_c(x)$$



Spreading with blocking:

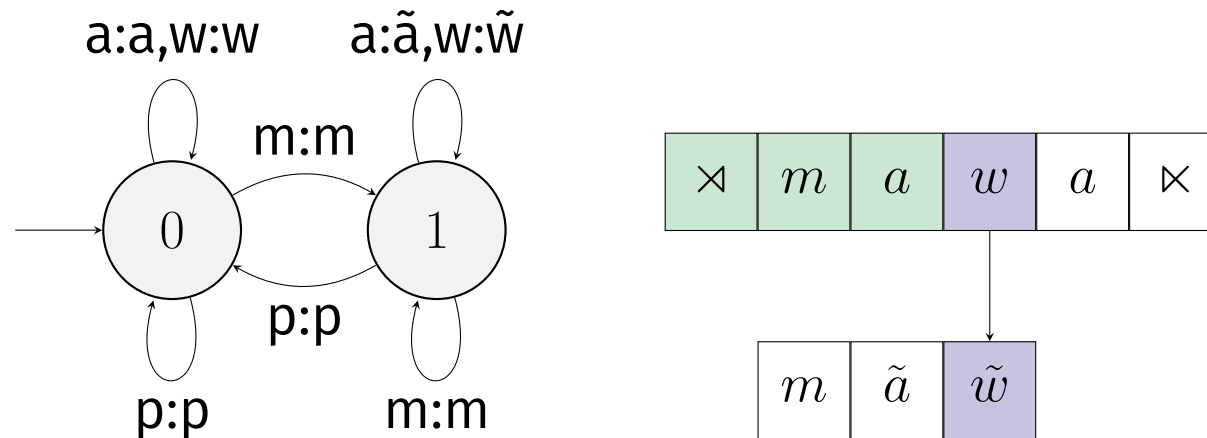
$$P'_b(x) \stackrel{\text{def}}{=} [\text{lfp}(P_b(x) \vee (A(p(x)) \wedge \neg P_c(\mathbf{x})))](x)$$

LD agreement:

$$P'_b(x) \stackrel{\text{def}}{=} [\text{lfp}(P_b(x) \vee A(p(x)))](x) \wedge \neg P_c(\mathbf{x})$$

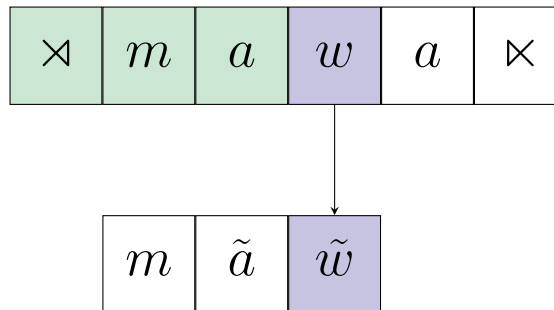
## QFLFP is (probably) subsequential

- **Subsequential functions** have some **deterministic** finite-state transducer (Schützenberger, 1977; Mohri, 1997)
- Reading left-to-right, we *immediately* know the output at each position in the input



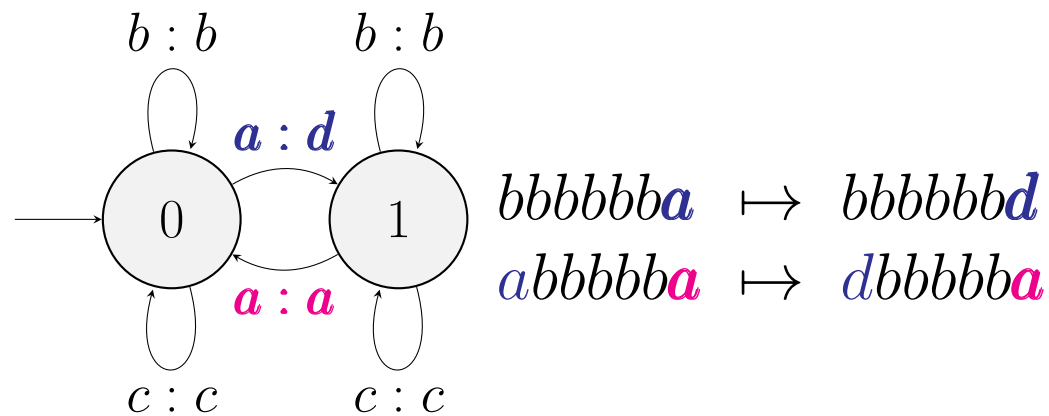
- For any  $\varphi(x) \in \text{QFLFP}$ , whether a position satisfies  $\varphi(x)$  depends entirely on the *preceding* information in the input
- Reading left-to-right, we *immediately* know the output at each position in the input

$$P_{\tilde{w}}(x) \stackrel{\text{def}}{=} [\text{lfp} + \text{son}(x) \wedge (P_m(p(x)) \vee A(p(x)))](x)$$



## Subsequential is (probably) not QFLFP

- Keeping track of even and odd-numbered elements of a particular type over arbitrary distances is subsequential



- We cannot think of a QFLFP definition for this function

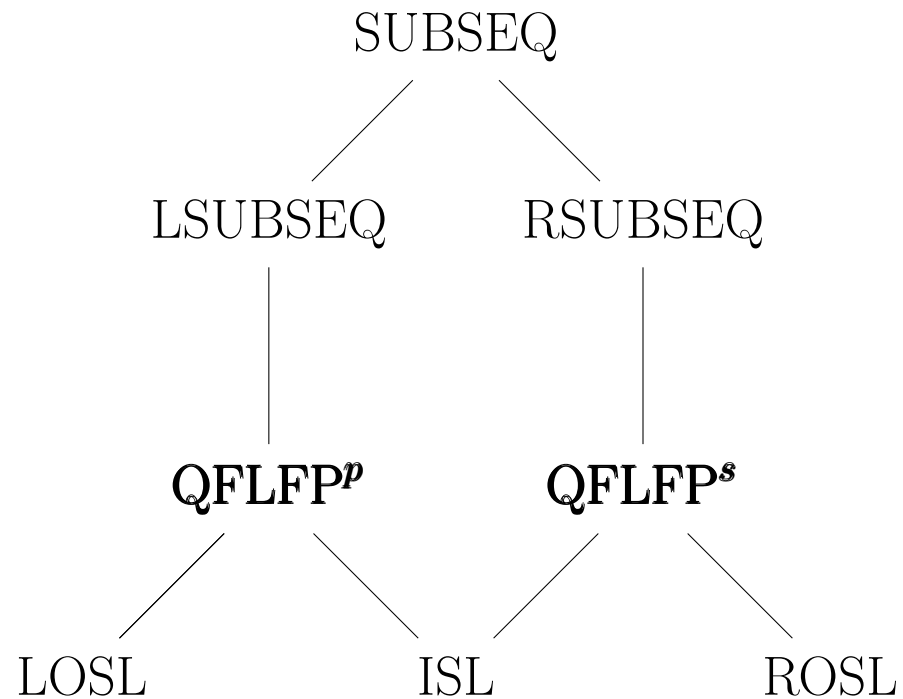
- This is a good *phonological* prediction of QFLFP; functions like “odd-numbered sibilants harmonize” are not attested.
- *But*, QFLFP can capture ‘local’ even/odd counting (for, e.g., iterative stress)

$$[\text{1fp } \times (p(x)) \vee A(p(p(x)))](x)$$

$\times_1$	$a_2$	$a_3$	$a_4$	$a_5$	$a_6$	$a_7$	$a_8$	$\times_9$
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## The general picture (probably)



OSL = output strictly local functions ([Chandlee, 2014](#); [Chandlee et al., 2015](#))

## Discussion

- QFLFP is a restrictive theory for phonology based on recursive definitions of local structures
- If  $\text{QFLFP} \subseteq \text{SUBSEQ}$ , then it is learnable (Oncina et al., 1993)
- Abstract definition of QFLFP?
- More efficient/plausible learner for QFLFP?

- Logic can be applied to non-string structures:
  - Features
  - Autosegmental representations
  - Metrical structure
  - Others?
- What do we get with two-place predicates and QFLFP (Koser et al., AMP)?

## Conclusion

- QFLFP combines the restrictiveness of QF with the ability to recursively reference the output structure.
- Allows us to model non-ISL phenomena such as LD agreement and iterative spreading.
- This class of functions appears to cross-cut several subregular classes that have been applied to the modeling of phonological processes.
- If/as a subset of subsequential, it is also learnable.

## **Acknowledgements**

Thanks to helpful thoughts and discussion from Jeff Heinz, Bill Idsardi, Steve Lindell, Jim Rogers, Jon Rawski, and the audience at the first Rutgers computational phonology workshop